Algebra in the Game *Lights Out*

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Abstract

The game *Lights Out* admits a natural formulation over the field $\mathbb{Z}_2$. This note studies $n \times n$ boards as binary matrices, describes solvable boards as the column space of a master click matrix, and relates the quotient by the solvable subgroup to the kernel of that matrix. The chasing-lights algorithm is interpreted as a linear reduction to the bottom row, giving a convenient upper bound for the number of equivalence classes of boards. The resulting top-row effect matrices exhibit a recursive parity structure, and their row spaces identify the solvable bottom-row representatives. Finally, Burnside's lemma is used to count boards, and solvable boards, up to the action of the dihedral group.

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1 Game Background

The classic version of *Lights Out* is an electronic handheld puzzle released by Tiger Electronics in 1995 [Wolb]. It is played on a 5×5 keypad of square buttons, each of which is either lit or dark. The object is to turn all lights off. Clicking a button changes its own state and the state of each of the four directly adjacent buttons when they exist. Thus a corner click changes three buttons, an edge click changes four buttons, and an interior click changes five buttons.

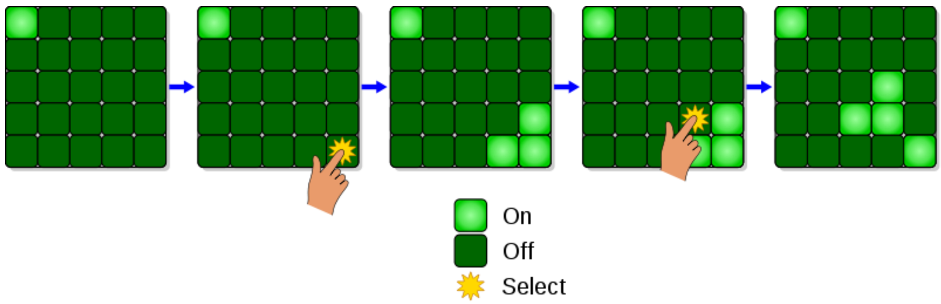


Figure 1: Sample play in *Lights Out*.

The mathematical literature surrounding this game predates the commercial puzzle. Sutner introduced the σ-game in the setting of cellular automata [Sut90]; the closely related σ^+-game is the graph-theoretic version in which pressing a vertex toggles that vertex and its neighbors. The usual square-board *Lights Out* puzzle is exactly this σ^+-game on a finite grid graph. This viewpoint connects the puzzle to reachability questions for additive cellular automata, including work on rectangular grids, cylinders, and tori by Barua and Ramakrishnan [BR96].

For the commercial 5×5 board, Anderson and Feil gave a concise linear-algebraic treatment over $\mathbb{Z}_2$, showing how solvability and solution counts are read from the associated click matrix [AF98]. Later work broadened this perspective in several directions. Hunziker, Machiavelo, and Park related reversibility on square grids to Chebyshev polynomials over finite fields [HMP04], which is the source of the rank-deficiency criterion used below. Kreh studied board-size and multi-state variants using linear algebra and algebraic number theory [Kre17]; Torrence isolated "easy" square and toroidal games, where pressing precisely the initially lit buttons solves the board [Tor11].

There is also a substantial graph-theoretic literature. Fleischer and Yu give a survey that unifies results from graph theory, recreational puzzle solving, and algorithms [FY13]. Berman, Borer, and Hungerbühler formulate *Lights Out* on general graphs using a Fredholm-type alternative and separating invariants [BBH21]. Related variants include lit-only games, in which only lit vertices may be played; these lead to minimum-light-number questions on grids and other graph classes [GWW11, Hua15]. The present note focuses on the classical square grid, but the same linear structure is what makes these wider connections possible.

We generalize to $n \times n$ boards. Such boards are the matrices in the abelian group

$$M_{n\times n}(\mathbb{Z}_2),$$

where addition is matrix addition modulo 2. Zeros represent dark squares and ones represent lit squares. The zero matrix is the dark board, and every board is its own inverse.

Let $[n] = \{1, \ldots, n\}$. Buttons are indexed by ordered pairs in $[n] \times [n]$.

2 Solvable Boards

Write $C_{i,j} \in M_{n\times n}(\mathbb{Z}_2)$ for the click matrix obtained by clicking the empty board at square $(i,j) \in [n] \times [n]$. If b is a board, then clicking (i,j) sends b to $b + C_{i,j}$. Since addition in $M_{n\times n}(\mathbb{Z}_2)$ is commutative, clicks may be performed in any order.

A board b is solvable if there is a subset $\Theta \subseteq [n] \times [n]$ such that

$$b + \sum_{(i,j)\in\Theta} C_{i,j} = 0.$$

Thus the subgroup of solvable boards is

$$\mathrm{Sol}_n = \langle C_{i,j} \mid (i,j) \in [n] \times [n] \rangle \le M_{n\times n}(\mathbb{Z}_2).$$

For a board b, let $\widehat{b} \in \mathbb{Z}_2^{n^2}$ denote the column vector formed by appending the rows of b. The *master click matrix*

$$\mathcal{M}_n = \begin{pmatrix} \widehat{C}_{1,1} & \cdots & \widehat{C}_{1,n} & \widehat{C}_{2,1} & \cdots & \widehat{C}_{2,n} & \cdots & \widehat{C}_{n,1} & \cdots & \widehat{C}_{n,n} \end{pmatrix}$$

has as its columns all possible click matrices.

Example 1. For a 3×3 board,

$$C_{1,1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widehat{C}_{1,1} = (1,1,0,1,0,0,0,0,0)^T.$$

The corresponding master click matrix is

$$\mathcal{M}_3 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The product $\mathcal{M}_n v$, where $v \in \mathbb{Z}_2^{n^2}$, selects and sums the columns of $\mathcal{M}_n$ corresponding to the nonzero coordinates of v. Therefore

$$\mathrm{col}(\mathcal{M}_n) = \mathrm{Sol}_n.$$

Solving a board b is equivalent to finding v such that $\mathcal{M}_n v = \widehat{b}$.

Proposition 2. *The following conditions are equivalent:*

(i) every $n \times n$ board is solvable;

(ii) $|\,\mathrm{Sol}_n\,| = 2^{n^2}$;

(iii) $\mathcal{M}_n$ has full rank over $\mathbb{Z}_2$;

(iv) $\mathcal{M}_n v = 0$ *has the unique solution* $v = 0$*;*

(v) $\det(\mathcal{M}_n) \neq 0$ *in* $\mathbb{Z}_2$*.*

Several OEIS sequences record related data:

- A075462: the number of solutions to $\mathcal{M}_n v = 0$;
- A076436: the values of n for which $\mathcal{M}_n v = 0$ has a unique solution;
- A117870: the values of n for which $\mathcal{M}_n v = 0$ has a nonunique solution;
- A159257: the rank deficiency of $\mathcal{M}_n$.

Slightly related are A075463, the number of solutions to $\mathcal{M}_n v = 0$ counted up to rotation and reflection, A076437, the values of n for which that solution is unique up to rotation and reflection, and A296350, the corresponding determinant data for a bordered *Lights Out* matrix, a related variant rather than the master matrix $\mathcal{M}_n$ used here.

2.1 Rank deficiency

Define the *rank deficiency*

$$\delta_n = \dim(\ker(\mathcal{M}_n)) = n^2 - \operatorname{rank}(\mathcal{M}_n).$$

The rank-deficiency criterion [HMP04] gives

$$\gamma_n(x) := \gcd(f_n(x), f_n(x+1)), \qquad \delta_n = \deg(\gamma_n),$$

where $f_n(x) = U_n(x/2)$ over $\mathbb{Z}_2$, and U_n is the nth Chebyshev polynomial of the second kind.

For the normalization used here, the Chebyshev polynomials of the second kind are characterized by

$$\sin((n+1)\theta) = U_n(\cos\theta)\sin\theta,$$

and satisfy the recursive definition

$$U_n(x) = \begin{cases} 1, & n = 0, \\ 2x, & n = 1, \\ 2xU_{n-1}(x) - U_{n-2}(x), & n \geq 2. \end{cases}$$

It follows that every $n \times n$ board is solvable precisely when $\delta_n = 0$, or equivalently when $f_n(x)$ and $f_n(x+1)$ share only a constant divisor.

Example 3. To find the rank deficiency of $\mathcal{M}_6$, compute

$$\begin{aligned} U_2(x) &= 2xU_1(x) - U_0(x) = 4x^2 - 1, \\ U_3(x) &= 2xU_2(x) - U_1(x) = 8x^3 - 4x, \\ U_4(x) &= 2xU_3(x) - U_2(x) = 16x^4 - 12x^2 + 1, \\ U_5(x) &= 2xU_4(x) - U_3(x) = 32x^5 - 32x^3 + 6x, \\ U_6(x) &= 2xU_5(x) - U_4(x) = 64x^6 - 80x^4 + 24x^2 - 1. \end{aligned}$$

n	$f_n(x)$	$f_n(x+1)$	$\gamma_n(x)$	δ_n
0	1	1	1	0
1	x	$x+1$	1	0
2	x^2+1	x^2	1	0
3	x^3	x^3+x^2+x+1	1	0
4	x^4+x^2+1	x^4+x^2+1	x^4+x^2+1	4
5	x^5+x	x^5+x^4	x^2+x	2
6	x^6+x^4+1	x^6+x^2+1	1	0
7	x^7	$x^7+x^6+x^5+x^4+x^3+x^2+x+1$	1	0
8	$x^8+x^6+x^4+1$	$x^8+x^6+x^2$	1	0
9	x^9+x^5+x	$x^9+x^8+x^5+x^4+x+1$	x^8+x^4+1	8
10	$x^{10}+x^8+x^4+x^2+1$	$x^{10}+x^4+1$	1	0
11	$x^{11}+x^3$	$x^{11}+x^{10}+x^9+x^8$	$x^6+x^5+x^4+x^3$	6
12	$x^{12}+x^{10}+x^8+x^2+1$	$x^{12}+x^{10}+x^8+x^4+1$	1	0
13	$x^{13}+x^9+x$	$x^{13}+x^{12}+x^5+x^4+x+1$	1	0
14	$x^{14}+x^{12}+x^8+1$	$x^{14}+x^{10}+x^8+x^6+x^2$	x^4+x^2+1	4

Table 1: The first rank-deficiency computations over $\mathbb{Z}_2$.

Thus

$$\begin{aligned} f_6(x) &= U_6(x/2) \\ &= x^6 - 5x^4 + 6x^2 - 1 \\ &\equiv x^6 + x^4 + 1 \pmod 2. \end{aligned}$$

Also

$$f_6(x+1) \equiv x^6 + x^2 + 1 \pmod 2.$$

Hence

$$\gamma_6(x) = \gcd(x^6+x^2+1, x^6+x^4+1) = 1, \qquad \delta_6 = \deg(\gamma_6) = 0.$$

Consequently every 6×6 board is solvable.

When $\mathcal{M}_n$ is not full rank, the solvable boards may still be generated by finding a basis for $\mathrm{col}(\mathcal{M}_n)$.

It is known that it is always possible to click the blank board to the all-ones board [Bro]. This is the all-ones problem. Since the diagonal of $\mathcal{M}_n$ is all ones, this is an application of the more general fact that the diagonal of any symmetric binary matrix lies in its column space.

Proposition 4. *The polynomials $f_n(x) = U_n(x/2)$ over $\mathbb{Z}_2$ are the Fibonacci polynomials with initial conditions*

$$f_0(x) = 1, \qquad f_1(x) = x, \qquad f_n(x) = xf_{n-1}(x) + f_{n-2}(x) \quad (n \geq 2).$$

Equivalently,

$$f_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^{n-2j} \pmod 2.$$

Proof. The Chebyshev polynomials of the second kind satisfy

$$U_n(z) = 2zU_{n-1}(z) - U_{n-2}(z).$$

Substituting $z = x/2$ gives

$$f_n(x) = x f_{n-1}(x) - f_{n-2}(x).$$

Over $\mathbb{Z}_2$, subtraction is addition, so this is precisely the Fibonacci polynomial recurrence. The displayed binomial formula follows by induction: the coefficient of x^{n-2j} in $x f_{n-1} + f_{n-2}$ is

$$\binom{n-1-j}{j} + \binom{n-1-j}{j-1} = \binom{n-j}{j}.$$

□

Proposition 5 (Inherited obstructions)**.** *If $m+1$ divides $n+1$, then*

$$f_m(x) \mid f_n(x) \qquad \textit{and} \qquad \gamma_m(x) \mid \gamma_n(x).$$

In particular, if $\delta_m > 0$, then $\delta_n > 0$.

Proof. Write $n+1 = ab$, where $a = m+1$. The sine identity for the Chebyshev polynomials of the second kind implies that $U_{a-1}(z)$ divides $U_{ab-1}(z)$ in $\mathbb{Z}[z]$: indeed,

$$U_{ab-1}(\cos\theta)\sin\theta = \sin(ab\theta)$$

is divisible by $\sin(a\theta) = U_{a-1}(\cos\theta)\sin\theta$. Substituting $z = x/2$ and reducing modulo 2 gives $f_m(x) \mid f_n(x)$. Replacing x by $x+1$ gives $f_m(x+1) \mid f_n(x+1)$. Therefore every common divisor of $f_m(x)$ and $f_m(x+1)$ is also a common divisor of $f_n(x)$ and $f_n(x+1)$, so $\gamma_m \mid \gamma_n$. □

This gives one natural notion of inheritance. Call a deficient size n *obstruction-primitive* if $\delta_n > 0$ and there is no smaller deficient $m < n$ with $m+1 \mid n+1$. Otherwise the rank deficiency of n contains an obstruction already present at a smaller board size. This notion is distinct from the effect-pattern genealogy below: a board size may be new for the 2-adic pattern tree while still carrying an inherited rank obstruction through the divisibility relation $m+1 \mid n+1$.

This distinction is useful because the pictures of the effect matrices and the solvability obstruction are measuring different kinds of structure. The divisibility relation explains when a smaller nullspace obstruction must reappear at a larger size, even when there is no obvious embedded copy of a smaller K_m in the visible effect matrix. The 2-adic genealogy below explains the embedded pictures themselves.

Proposition 6 (Odd doubling identity)**.** *For every $n \geq 0$,*

$$f_{2n+1}(x) = x f_n(x)^2$$

over $\mathbb{Z}_2$. Consequently, if $\delta_n \neq 0$, then $\delta_{2n+1} \neq 0$.

Proof. The identity follows from the binomial formula and Lucas' theorem: modulo 2, the nonzero terms of f_{2n+1} occur exactly when the summation index is even, and the resulting expression is

$$x \left(\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^{n-2j} \right)^2.$$

If $\delta_n \neq 0$, then $f_n(x)$ and $f_n(x+1)$ have a common nonconstant divisor $h(x)$. The identity above gives

$$f_{2n+1}(x) = x f_n(x)^2, \qquad f_{2n+1}(x+1) = (x+1) f_n(x+1)^2,$$

so $h(x)$ also divides both $f_{2n+1}(x)$ and $f_{2n+1}(x+1)$. The rank-deficiency criterion then implies $\delta_{2n+1} \neq 0$. □

Proposition 7 (Solvability criterion)**.** *A board $b \in M_{n\times n}(\mathbb{Z}_2)$ is solvable if and only if*

$$\langle \widehat{b}, q\rangle = 0 \quad \textit{for every } q \in \ker(\mathcal{M}_n),$$

where $\langle u, v\rangle = \sum_i u_i v_i$ is the standard dot product over $\mathbb{Z}_2$.

Proof. The linear algebra statement behind this is the Fredholm alternative over $\mathbb{Z}_2$:

$$\operatorname{im}(A) = (\ker(A^T))^{\perp}.$$

It is used in this form for *Lights Out* on graphs by Berman, Borer, and Hungerbühler [BBH21]. Since $\mathcal{M}_n$ is symmetric, $\ker(\mathcal{M}_n^T) = \ker(\mathcal{M}_n)$. Also $\mathrm{Sol}_n = \operatorname{col}(\mathcal{M}_n)$. Therefore

$$b \in \mathrm{Sol}_n \iff \widehat{b} \in \operatorname{im}(\mathcal{M}_n) \iff \widehat{b} \perp \ker(\mathcal{M}_n).$$

□

Proposition 8. *If $\mathcal{M}_n$ is invertible, then there is an invertible binary matrix X such that $X^T X = \mathcal{M}_n$. This means that the symmetric bilinear form represented by $\mathcal{M}_n$ is congruent to the standard dot product over $\mathbb{Z}_2$.*

Proof. The matrix $\mathcal{M}_n$ is symmetric and has nonzero diagonal. When it is invertible, it represents a nondegenerate nonalternating symmetric bilinear form over $\mathbb{Z}_2$. Such a form has an orthonormal basis: choose a vector v with $\langle v, v\rangle_{\mathcal{M}_n} = 1$, split off its orthogonal complement, and continue by induction. In that basis the Gram matrix is the identity. If P is the matrix whose columns are this orthonormal basis, then $P^T \mathcal{M}_n P = I$, so with $X = P^{-1}$ we have $\mathcal{M}_n = X^T X$. This is the content of the general matrix result cited in [Mat]; it gives a congruence classification rather than an additional *Lights Out*- specific solvability condition. □

3 The Quotient Group Λ_n

Since $M_{n\times n}(\mathbb{Z}_2)$ is abelian, Sol_n is normal, so we may define

$$\Lambda_n = M_{n\times n}(\mathbb{Z}_2)/\,\mathrm{Sol}_n\,.$$

Let $b_1 \sim_\lambda b_2$ when b_1 and b_2 lie in the same coset of Sol_n. Then

$$\begin{aligned} b_1 \sim_\lambda b_2 &\iff b_1 + \mathrm{Sol}_n = b_2 + \mathrm{Sol}_n \\ &\iff b_1 - b_2 \in \mathrm{Sol}_n \\ &\iff b_1 - b_2 = \sum_{(i,j)\in\Theta} C_{i,j} \quad \text{for some } \Theta \subseteq [n]\times[n] \\ &\iff b_1 + \sum_{(i,j)\in\Theta} C_{i,j} = b_2. \end{aligned}$$

Thus two boards represent the same element of Λ_n exactly when one can be clicked to the other.

The size of Λ_n measures how solvable the whole board space is. A small Λ_n means there are few cosets of Sol_n, so Sol_n is large inside $M_{n\times n}(\mathbb{Z}_2)$; a large Λ_n means many distinct click-equivalence classes.

From $\mathrm{Sol}_n = \operatorname{col}(\mathcal{M}_n)$ and rank-nullity,

$$\dim(\operatorname{col}(\mathcal{M}_n)) = n^2 - \delta_n.$$

Therefore

$$|\operatorname{col}(\mathcal{M}_n)| = 2^{n^2-\delta_n},$$

and hence

$$|\Lambda_n| = \frac{|M_{n\times n}(\mathbb{Z}_2)|}{|\operatorname{col}(\mathcal{M}_n)|} = 2^{\delta_n} = |\ker(\mathcal{M}_n)|.$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\|\Lambda_n\|$	1	1	1	16	4	1	1	1	256	1	64	1	1	16	1	256	4	1	65536

Table 2: The first values of $|\Lambda_n|$.

Proposition 9. *The number $|\Lambda_n| = |\ker(\mathcal{M}_n)|$ is also the number of ways to click the empty board to the all-ones board.*

Proof. Let $r = \dim(\operatorname{col}(\mathcal{M}_n))$. Reorder columns, if necessary, so that the first r columns form a basis for $\operatorname{col}(\mathcal{M}_n)$. Since the all-ones board $\mathbf{1}$ lies in $\operatorname{col}(\mathcal{M}_n)$, there is a unique $v \in \mathbb{Z}_2^{n^2}$ such that $\mathcal{M}_n v = \mathbf{1}$ and the last $n^2 - r$ entries of v are zero. For every $w \in \ker(\mathcal{M}_n)$,

$$\mathcal{M}_n(v+w) = \mathcal{M}_n v + \mathcal{M}_n w = \mathbf{1}.$$

Conversely, any solution differs from v by an element of $\ker(\mathcal{M}_n)$. Thus there are $|\ker(\mathcal{M}_n)| = |\Lambda_n|$ such click patterns. $\square$

Proposition 10. *Let $q_1, \dots, q_d$ be a basis for $\ker(\mathcal{M}_n)$, where $d = \delta_n$. The map*

$$\begin{aligned} \Phi : \Lambda_n &\longrightarrow \mathbb{Z}_2^d \\ b + \operatorname{Sol}_n &\longmapsto \left(\langle \widehat{b}, q_1\rangle, \dots, \langle \widehat{b}, q_d\rangle\right) \end{aligned}$$

is an isomorphism. In particular,

$$\Lambda_n \cong \mathbb{Z}_2^{\delta_n}.$$

Proof. The solvability criterion above shows that $\widehat{s} \perp q_i$ for every $s \in \operatorname{Sol}_n$ and every i. Therefore Φ is well defined on cosets. It is a homomorphism because the dot product is linear. Its kernel consists of those cosets $b + \operatorname{Sol}_n$ for which $\widehat{b} \perp \ker(\mathcal{M}_n)$, which is equivalent to $b \in \operatorname{Sol}_n$. Hence the kernel is trivial. Since

$$|\Lambda_n| = 2^{\delta_n} = |\mathbb{Z}_2^d|,$$

the injective homomorphism Φ is an isomorphism. $\square$

4 Chasing Lights and Effect Matrices

Let $E_{i,j}$ denote the board with only the square (i,j) lit. Define

$$L_n = \langle E_{n,i} \mid i \in [n]\rangle \leq M_{n\times n}(\mathbb{Z}_2),$$

the subgroup generated by boards with lights only in the bottom row. Clearly $L_n \cong \mathbb{Z}_2^n$.

The chasing-lights algorithm is a function

$$\operatorname{CL} : M_{n\times n}(\mathbb{Z}_2) \to L_n.$$

To perform it, iterate through each button in the first $n-1$ rows, starting in the top left and processing horizontally. If the current button is lit, click the button directly below it. The result has lights only in the bottom row.

Proposition 11. *The chasing-lights map* $\mathrm{CL} : M_{n\times n}(\mathbb{Z}_2) \to L_n$ *is a surjective group homomorphism.*

Proof. Surjectivity is immediate because a board already supported on the bottom row is fixed by the algorithm. For linearity, view a board as a vector in $\mathbb{Z}_2^{n^2}$. At each step (i, j), with $i < n$, the algorithm applies the map

$$x \longmapsto x + x_{i,j} C_{i+1,j}.$$

This map is linear over $\mathbb{Z}_2$, and CL is a composition of these linear maps followed by projection to the resulting bottom-row board. Hence $\mathrm{CL}(b + c) = \mathrm{CL}(b) + \mathrm{CL}(c)$. □

Lemma 12. *If* (i, j) *is a button pressed during the chasing-lights algorithm on board* b*, then*

$$\mathrm{CL}(b) = \mathrm{CL}(b + C_{i,j}).$$

Proof. Since the chasing-lights algorithm is deterministic, let Θ be the ordered set of buttons pressed during the algorithm on b. Choose $(i, j) \in \Theta$. Running the algorithm on $b + C_{i,j}$ is the same as moving the click (i, j) to the beginning of the click sequence. The buttons before (i, j) are determined by entries above them, which are unchanged by this reordering, and after (i, j) has been applied the two algorithm states agree. Since clicks commute, the final bottom-row board is the same. □

Proposition 13. *The chasing-lights map gives representatives for the cosets of* Sol_n *in the bottom-row space* L_n*. More precisely,*

$$\Lambda_n \cong L_n/(L_n \cap \mathrm{Sol}_n),$$

so a row-reduced choice of representatives for the quotient $L_n/(L_n \cap \mathrm{Sol}_n)$ *gives a representative set for* Λ_n*. In particular,* $\dim \Lambda_n \le n$.

Proof. Every board $b \in M_{n\times n}(\mathbb{Z}_2)$ is click-equivalent to $\mathrm{CL}(b) \in L_n$, because the chasing-lights algorithm is itself a sequence of clicks. Thus the natural map

$$g : L_n \longrightarrow \Lambda_n, \qquad b \longmapsto b + \mathrm{Sol}_n$$

is surjective. Its kernel is

$$\ker(g) = \{b \in L_n \mid b \in \mathrm{Sol}_n\} = L_n \cap \mathrm{Sol}_n\,.$$

The first isomorphism theorem gives $\Lambda_n \cong L_n/(L_n \cap \mathrm{Sol}_n)$. This is the bottom-row version of the standard light-chasing reduction described by Scherphuis [Sch]. Since $L_n \cong \mathbb{Z}_2^n$, the quotient has dimension at most n. □

Proposition 14 (Top-row reduction)**.** *If* $b \in \mathrm{Sol}_n$*, then there is some*

$$b' \in \langle C_{1,i} \mid i \in [n]\rangle$$

such that $\mathrm{CL}(b') = \mathrm{CL}(b)$.

Proof. Since $b \in \mathrm{Sol}_n$, there is a click pattern $v \in \mathbb{Z}_2^{n^2}$ with $\mathcal{M}_n v = \widehat{b}$. Split $v = t + w$, where t is supported on the first row and w is supported on rows $2, \ldots, n$. Let b' be the board corresponding to the first-row clicks t, so $\widehat{b'} = \mathcal{M}_n t$.

It remains to show that $\mathrm{CL}(\mathcal{M}_n w) = 0$. When a click pattern has no first-row clicks, the first row of the board $\mathcal{M}_n w$ records exactly the second row of the click pattern w. Therefore the first step of chasing lights clicks precisely those second-row buttons. After those clicks are made, the same argument applies one row lower: the next lit row records the next row of w. Inductively, the chasing algorithm performs exactly the clicks represented by w, and since $\mathcal{M}_n w$ was produced by those clicks, the final bottom row is zero. Hence

$$\mathrm{CL}(b) = \mathrm{CL}(\mathcal{M}_n(t+w)) = \mathrm{CL}(\mathcal{M}_n t) = \mathrm{CL}(b').$$

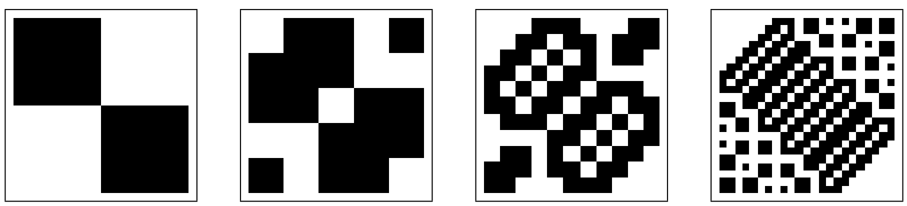
This is the algebraic form of the standard first-row light-chasing method [Sch]. □

For $n \geq 1$, define the *top-row effect matrix* K_n by declaring that its ith row is the bottom row of $\mathrm{CL}(C_{1,i})$. Equivalently, the transpose of K_n is the matrix of the linear map

$$\langle C_{1,i} \mid i \in [n] \rangle \longrightarrow L_n, \qquad b \longmapsto \mathrm{CL}(b),$$

with respect to the evident bases. Neller's pictures display these matrices as $n \times n$ black-and-white images [Nel26].

The point of K_n is that it records all possible effects of first-row click choices after chasing. The top-row reduction proposition says that, for solvable boards, these first-row effects already account for every solvable bottom-row representative.



K_2 K_5 K_{11} K_{23}

Figure 2: Top-row effect patterns provided by Todd Neller [Nel26]. Each pattern reappears in the next one on the even-numbered rows and columns.

Let B_n be the $n \times n$ tridiagonal matrix over $\mathbb{Z}_2$ with ones on the diagonal and on the two adjacent diagonals. If $u \in \mathbb{Z}_2^n$ is a top-row click pattern and p_t is the row t click pattern generated by chasing, with $p_0 = 0$ and $p_1 = u$, then

$$p_{t+1} = B_n p_t + p_{t-1}.$$

Thus $p_{n+1} = f_n(B_n)u$, where f_n is the Fibonacci polynomial from the rank-deficiency section. Hence

$$K_n = f_n(B_n).$$

This row-recurrence viewpoint is standard in the parity-domination literature on grid graphs [GKW02, KG02].

Definition 15. After identifying L_n with $\mathbb{Z}_2^n$, define the *solvable bottom-row subspace*

$$\mathcal{C}_n = L_n \cap \mathrm{Sol}_n\,.$$

Proposition 16 (The solvable bottom-row subspace)**.** *The solvable bottom-row subspace is the row space of the top-row effect matrix:*

$$\mathcal{C}_n = \mathrm{rowspan}(K_n).$$

Consequently,

$$\dim(\mathcal{C}_n) = \mathrm{rank}(K_n) = n - \delta_n.$$

Proof. The inclusion $\mathrm{rowspan}(K_n) \subseteq L_n \cap \mathrm{Sol}_n$ holds because each row of K_n is $\mathrm{CL}(C_{1,i})$, and chasing changes a board by a sequence of clicks. Conversely, if $c \in L_n \cap \mathrm{Sol}_n$, then $c = \mathrm{CL}(c)$. By the top-row reduction proposition there is a sum b' of first-row click matrices with $\mathrm{CL}(b') = \mathrm{CL}(c) = c$. Therefore c is a sum of rows of K_n.

For the dimension formula, use the earlier isomorphism $\Lambda_n \cong L_n/(L_n \cap \mathrm{Sol}_n)$. Since $\dim L_n = n$ and $\dim \Lambda_n = \delta_n$, we get

$$\dim(\mathcal{C}_n) = n - \delta_n.$$

The equality $\dim(\mathcal{C}_n) = \mathrm{rank}(K_n)$ follows from $\mathcal{C}_n = \mathrm{rowspan}(K_n)$. □

Theorem 17 (Self-embedding of effect patterns)**.** For every $n \geq 1$, the top-row effect matrix K_n occurs as the even-row, even-column submatrix of K_{2n+1}. That is,

$$(K_{2n+1})_{2i,2j} = (K_n)_{i,j} \qquad (1 \leq i, j \leq n).$$

This theorem is interesting because it turns a visible pattern in the pictures into an exact self-similarity statement. The smaller effect matrix does not merely resemble part of the larger one; it appears literally on a parity sublattice.

Proof. Let $B = B_{2n+1}$. From the Fibonacci-polynomial identity proved above,

$$f_{2n+1}(x) = x f_n(x)^2$$

over $\mathbb{Z}_2$. Therefore

$$K_{2n+1} = f_{2n+1}(B) = B f_n(B)^2 = B f_n(B^2).$$

The matrix B^2 has no entries between an odd index and an even index: the two length-two routes between adjacent parities cancel modulo 2. On the even-numbered indices $2, 4, \ldots, 2n$, the matrix B^2 restricts to B_n: from $2i$ there is one contribution to itself and one to each of $2i-2$ and $2i+2$, when those indices exist. Thus the even-even block of $f_n(B^2)$ is $f_n(B_n) = K_n$.

Finally, the even-even block of B itself is the identity matrix: from an even index, the only same-parity closed-neighborhood move is the loop. Multiplication by the leading B therefore leaves the even-even block equal to K_n, proving the claim. □

Proposition 18 (The remaining rows of K_{2n+1})**.** *Let J_n be the $(n+1) \times n$ matrix over $\mathbb{Z}_2$ defined by*

$$(J_n)_{p,j} = 1 \quad \textit{if and only if} \quad j = p-1 \text{ or } j = p,$$

where indices outside $[n]$ are ignored. Thus

$$J_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Let D_1 and D_{n+1} be the $(n+1) \times (n+1)$ diagonal matrices with a single nonzero entry in positions $(1,1)$ and $(n+1, n+1)$, and put

$$R_{n+1} = B_{n+1} + D_1 + D_{n+1}, \qquad H_n = f_n(R_{n+1}).$$

If K_{2n+1} is written with odd-indexed rows and columns first, followed by even-indexed rows and columns, then

$$K_{2n+1} \sim \begin{pmatrix} H_n & J_n K_n \\ K_n J_n^T & K_n \end{pmatrix}.$$

In particular, for $1 \le p \le n+1$ and $1 \le j \le n$,

$$(K_{2n+1})_{2p-1,2j} = (K_n)_{p-1,j} + (K_n)_{p,j},$$

where the rows 0 and $n+1$ of K_n are interpreted as zero rows.

Proof. Let $B = B_{2n+1}$. After ordering indices as

$$1, 3, 5, \ldots, 2n+1, 2, 4, \ldots, 2n,$$

the matrix B has block form

$$B = \begin{pmatrix} I_{n+1} & J_n \\ J_n^T & I_n \end{pmatrix}.$$

Squaring gives

$$B^2 = \begin{pmatrix} I_{n+1} + J_n J_n^T & 0 \\ 0 & I_n + J_n^T J_n \end{pmatrix}.$$

The lower-right block is B_n. The upper-left block is B_{n+1} with its two endpoint diagonal entries toggled, namely R_{n+1}. Hence

$$f_n(B^2) = \begin{pmatrix} H_n & 0 \\ 0 & K_n \end{pmatrix}.$$

Using $f_{2n+1}(x) = x f_n(x)^2$, we have

$$K_{2n+1} = f_{2n+1}(B) = B f_n(B^2),$$

so the displayed block form follows. Since K_{2n+1} is symmetric, the lower-left block may be written as $K_n J_n^T$. The entrywise formula is the statement that row p of $J_n K_n$ is the sum of rows $p-1$ and p of K_n, with nonexistent rows treated as zero. □

Figure 3 shows this decomposition for the first few terms in the $K_2, K_5, K_{11}, K_{23}, \ldots$ family. The off-diagonal blocks make the action of J_n visible: on the even columns, each odd-indexed row is obtained by adding two adjacent rows of the embedded copy of K_n.

Thus the rows of K_{2n+1} not belonging to the embedded copy of K_n are still controlled by K_n on the even columns: they are adjacent row sums of K_n. The genuinely new part is the odd-indexed block H_n, which is governed by the same Fibonacci polynomial evaluated on a path with modified endpoint behavior.

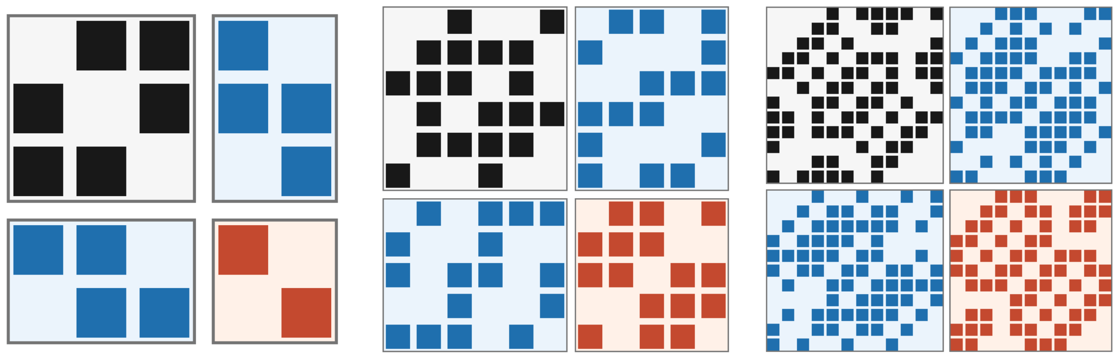
This answers a natural question raised by the pictures: the rows not inherited from the smaller K_n are not arbitrary. Off the new odd-indexed block, they are discrete derivatives of adjacent old rows, and only H_n carries new endpoint data.

Iterating $n \mapsto 2n+1$ from $n = 2$ gives

$$2,\ 5,\ 11,\ 23,\ 47, \ldots, \qquad n = 3 \cdot 2^k - 1.$$

This is the restart sequence visible in Figure 2: each new pattern contains the previous pattern on a fixed parity sublattice.

More generally, the preceding theorem is only one instance of a larger 2-adic structure. Call K_r a *root pattern* for K_n if $n = (r+1)2^t - 1$ for some $t \ge 0$, with $r = 1$ or r even. Equivalently, r is obtained from n by repeatedly applying $n \mapsto (n-1)/2$ while the current index is odd, stopping at 1 or at an even number.



K_5 $\qquad$ K_{11} $\qquad$ K_{23}

Figure 3: The matrices K_{2n+1} with rows and columns ordered as $1, 3, \ldots, 2n+1, 2, 4, \ldots, 2n$. Dark cells are the upper-left block H_n, blue cells are the off-diagonal blocks $J_n K_n$ and $K_n J_n^T$, and orange cells are the lower-right embedded copy of K_n.

Proposition 19 (Genealogy of effect patterns)**.** *Let $r \geq 1$ and $t \geq 0$, and put*

$$n = (r+1)2^t - 1.$$

Then K_r appears inside K_n on the rows and columns indexed by

$$2^t,\ 2 \cdot 2^t,\ \ldots,\ r \cdot 2^t.$$

That is,

$$(K_n)_{2^t i,\, 2^t j} = (K_r)_{i,j} \qquad (1 \leq i, j \leq r).$$

This proposition packages the self-embedding theorem into a family tree for effect matrices. It gives a precise meaning to the idea that a visible pattern has an ancestor, while also making clear that this is a 2-adic notion of ancestry rather than the same thing as rank-deficiency inheritance.

Proof. The case $t = 0$ is immediate. The self-embedding theorem proves the case $t = 1$. Iterating that theorem gives the result: each passage from m to $2m+1$ doubles the row and column indices of the embedded copy. $\square$

$$\begin{array}{ccccccccc}
K_1 & \longrightarrow & K_3 & \longrightarrow & K_7 & \longrightarrow & K_{15} & \longrightarrow & \cdots \\
K_2 & \longrightarrow & K_5 & \longrightarrow & K_{11} & \longrightarrow & K_{23} & \longrightarrow & \cdots \\
K_4 & \longrightarrow & K_9 & \longrightarrow & K_{19} & \longrightarrow & K_{39} & \longrightarrow & \cdots \\
K_6 & \longrightarrow & K_{13} & \longrightarrow & K_{27} & \longrightarrow & K_{55} & \longrightarrow & \cdots \\
K_8 & \longrightarrow & K_{17} & \longrightarrow & K_{35} & \longrightarrow & K_{71} & \longrightarrow & \cdots
\end{array}$$

Figure 4: A finite window of the effect-pattern family forest. Each arrow denotes the embedding $K_m \subset K_{2m+1}$.

Thus every K_n belongs to a unique family whose root is either K_1 or an even-indexed pattern. These roots are primitive only with respect to the even-sublattice embedding above; they may still carry inherited rank obstructions in the sense of $m+1 \mid n+1$. The even roots nevertheless have a strong internal form.

The next result explains a second visual phenomenon: even roots can look like an interpolation between their two neighboring odd sizes. Over $\mathbb{Z}_2$ this is not an average but a superposition: matrix addition over $\mathbb{Z}_2$, which is entrywise XOR of the black-and-white effect pictures. The polynomial f_{2m} splits as the sum of two squares, so the effect matrix K_{2m} is assembled by XORing the two half-scale evolutions governed by f_m and f_{m-1}.

Proposition 20 (Even-root superposition)**.** *Let $n = 2m$. Then K_{2m} has no entries connecting opposite parities:*

$$(K_{2m})_{i,j} = 0 \qquad \textit{whenever } i \not\equiv j \pmod 2.$$

Moreover,

$$K_{2m} = f_m(B_{2m}^2) + f_{m-1}(B_{2m}^2).$$

The addition in this formula is entrywise addition modulo 2, hence entrywise XOR of the two displayed patterns. More precisely, let D_1 and D_m be the $m \times m$ diagonal matrices with a single nonzero entry in positions $(1,1)$ and (m,m), respectively. If $h_m(x) = f_m(x) + f_{m-1}(x)$, then the odd-indexed block of K_{2m} is

$$h_m(B_m + D_1),$$

and the even-indexed block is

$$h_m(B_m + D_m).$$

Proof. The Fibonacci addition identity

$$f_{a+b}(x) = f_a(x)f_b(x) + f_{a-1}(x)f_{b-1}(x)$$

follows by induction from the recurrence for f_n. Taking $a = b = m$ gives

$$f_{2m}(x) = f_m(x)^2 + f_{m-1}(x)^2.$$

Therefore, with $B = B_{2m}$,

$$K_{2m} = f_{2m}(B) = f_m(B^2) + f_{m-1}(B^2) = h_m(B^2).$$

The matrix B^2 preserves parity, since the two length-two routes from an index to a neighboring opposite-parity index cancel modulo 2. Hence every polynomial in B^2, and in particular K_{2m}, has zero off-parity blocks.

It remains only to identify the two parity blocks of B^2. On the odd indices $1, 3, \ldots, 2m-1$, the restriction is the tridiagonal matrix B_m with its first diagonal entry toggled, namely $B_m + D_1$. On the even indices $2, 4, \ldots, 2m$, it is B_m with its last diagonal entry toggled, namely $B_m + D_m$. Applying h_m to these two restricted blocks gives the displayed formulas. $\square$

Example 21. The first nontrivial self-embedding occurs in K_5. Ordering rows and columns as $1, 3, 5, 2, 4$, one obtains

$$K_5 \sim \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} H_2 & J_2K_2 \\ K_2J_2^T & K_2 \end{pmatrix}.$$

Here

$$K_2 = I_2, \qquad J_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus the lower-right block is the inherited copy of K_2, while the upper-right block consists of the first row of K_2, the sum of the two rows of K_2, and the second row of K_2. This is the smallest place where the non-inherited rows can be seen as adjacent row sums rather than new random data.

The first even-root behavior is already visible in K_6. Ordering rows and columns as $1, 3, 5, 2, 4, 6$, one gets

$$K_6 \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The two off-diagonal parity blocks are zero, illustrating the parity splitting of even roots. The two diagonal blocks are the two halves combined by

$$K_6 = f_3(B_6^2) + f_2(B_6^2),$$

where the addition is entrywise XOR.

5 Boards and Symmetries

5.1 Orbits of all boards

The dihedral group D_4, the symmetry group of the square, acts naturally on $M_{n\times n}(\mathbb{Z}_2)$. Burnside's lemma counts the number $N(n)$ of possible board states up to rotation and reflection.

If n is even, an $n \times n$ board is made of concentric square layers of sizes $4(n-1), 4(n-3), \ldots, 4$. The identity fixes 2^{n^2} boards. The two rotations by $\pi/2$ fix $2^{n^2/4}$ boards each. The rotation by π fixes $2^{n^2/2}$ boards. The horizontal and vertical reflections fix $2^{n^2/2}$ boards each, while the two diagonal reflections fix $2^{n(n+1)/2}$ boards each.

If n is odd, the two rotations by $\pi/2$ fix $2^{(n^2+3)/4}$ boards each. The rotation by π fixes $2^{(n^2+1)/2}$ boards. Each reflection fixes $2^{n(n+1)/2}$ boards. Therefore

$$N(n) = \begin{cases} \frac{1}{8}\left(2^{n^2} + 2\cdot 2^{n^2/4} + 2^{n^2/2} + 2\cdot 2^{n^2/2} + 2\cdot 2^{n(n+1)/2}\right), & n \equiv 0 \pmod 2, \\ \frac{1}{8}\left(2^{n^2} + 2\cdot 2^{(n^2+3)/4} + 2^{(n^2+1)/2} + 4\cdot 2^{n(n+1)/2}\right), & n \equiv 1 \pmod 2. \end{cases}$$

Equivalently,

$$N(n) = \begin{cases} 2^{n^2-3} + 2^{(n^2-8)/4} + 2^{(n^2-6)/2} + 2^{(n^2-4)/2} + 2^{(n(n+1)-4)/2}, & n \equiv 0 \pmod 2, \\ 2^{n^2-3} + 2^{(n^2-5)/4} + 2^{(n^2-5)/2} + 2^{(n^2+n-2)/2}, & n \equiv 1 \pmod 2. \end{cases}$$

Proposition 22 (Asymptotics for all board orbits)**.** *As $n \to \infty$,*

$$N(n) = 2^{n^2-3}\left(1 + O\left(2^{-(n^2-n)/2}\right)\right).$$

In particular,

$$N(n) \sim \frac{2^{n^2}}{8}.$$

Proof. Burnside's formula writes $N(n)$ as $1/8$ times the sum of the numbers of boards fixed by the eight elements of D_4. The identity fixes all 2^{n^2} boards. Every nonidentity symmetry fixes at most

$$2^{n(n+1)/2}$$

boards, since the largest fixed spaces come from reflections. Therefore the seven nonidentity terms contribute

$$O\left(2^{n(n+1)/2}\right),$$

and hence

$$N(n) = \frac{1}{8}\left(2^{n^2} + O\left(2^{n(n+1)/2}\right)\right) = 2^{n^2-3}\left(1 + O\left(2^{-(n^2-n)/2}\right)\right).$$

□

Equivalently, the proportion of boards with a nontrivial square symmetry is exponentially small. Thus almost every D_4-orbit has size 8, and the main term $2^{n^2}/8$ is simply the total number of boards divided by the order of the symmetry group.

n	1	2	3	4	5	6
$N(n)$	2	6	102	8548	4211744	8590557312

Table 3: The first values of $N(n)$.

This sequence appears as OEIS A054247.

Example 23. For $n = 2$, the six D_4-orbits in $M_{2\times 2}(\mathbb{Z}_2)$ are

$$\begin{array}{c|l}
1 & \begin{pmatrix}0&0\\0&0\end{pmatrix} \\
2 & \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix} \\
3 & \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix} \\
4 & \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix} \\
5 & \begin{pmatrix}0&1\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix} \\
6 & \begin{pmatrix}1&1\\1&1\end{pmatrix}.
\end{array}$$

This agrees with $N(2) = 6$.

5.2 Orbits of solvable boards

The count above treats all 2^{n^2} boards. Since a square symmetry sends the click at (i, j) to the click at the transformed square, the action of D_4 also preserves the solvable subgroup Sol_n.

For $g \in D_4$, let

$$\mathrm{Fix}(g) = \{b \in M_{n\times n}(\mathbb{Z}_2) \mid g \cdot b = b\}.$$

Theorem 24 (Burnside count for solvable boards)**.** The number $N_{\mathrm{Sol}}(n)$ of solvable $n \times n$ boards up to rotation and reflection is

$$N_{\mathrm{Sol}}(n) = \frac{1}{8} \sum_{g \in D_4} |\operatorname{Sol}_n \cap \operatorname{Fix}(g)|.$$

Equivalently, if

$$d_g = \dim_{\mathbb{Z}_2}(\operatorname{Sol}_n \cap \operatorname{Fix}(g)),$$

then

$$N_{\mathrm{Sol}}(n) = \frac{1}{8} \sum_{g \in D_4} 2^{d_g}.$$

Proof. For every $g \in D_4$ and every click position $(i, j) \in [n] \times [n]$,

$$g \cdot C_{i,j} = C_{g(i,j)}.$$

Thus g permutes the generators of Sol_n, so $g \cdot \operatorname{Sol}_n = \operatorname{Sol}_n$. Therefore D_4 acts on the finite set Sol_n. Burnside's lemma gives the number of orbits as the average number of solvable boards fixed by a group element:

$$|\operatorname{Sol}_n / D_4| = \frac{1}{|D_4|} \sum_{g \in D_4} |\{b \in \operatorname{Sol}_n \mid g \cdot b = b\}|.$$

Since $|D_4| = 8$ and the fixed set inside Sol_n is precisely $\operatorname{Sol}_n \cap \operatorname{Fix}(g)$, this proves the first formula. The intersection $\operatorname{Sol}_n \cap \operatorname{Fix}(g)$ is a vector subspace of $M_{n\times n}(\mathbb{Z}_2)$ over $\mathbb{Z}_2$, so its cardinality is 2^{d_g}, which gives the second formula. □

The preceding formula becomes more explicit after separating the ordinary fixed-space count from the kernel obstructions. Let

$$\mathcal{K}_n = \ker(\mathcal{M}_n), \qquad F_g = \operatorname{Fix}(g), \qquad f_g(n) = \dim(F_g).$$

Define the *symmetry correction*

$$\eta_g(n) = \dim\big(\mathcal{K}_n \cap F_g^{\perp}\big).$$

Equivalently, since g acts by an orthogonal permutation matrix,

$$F_g^{\perp} = \operatorname{im}(g - I),$$

so

$$\eta_g(n) = \dim\big(\mathcal{K}_n \cap \operatorname{im}(g - I)\big).$$

Proposition 25 (Closed Burnside formula for solvable boards)**.** *For each* $g \in D_4$*,*

$$|\operatorname{Sol}_n \cap \operatorname{Fix}(g)| = 2^{f_g(n) - \delta_n + \eta_g(n)}.$$

Consequently,

$$N_{\mathrm{Sol}}(n) = \frac{1}{8} \sum_{g \in D_4} 2^{f_g(n) - \delta_n + \eta_g(n)}.$$

Proof. Since $\mathcal{M}_n$ is symmetric,

$$\mathrm{Sol}_n = \mathrm{col}(\mathcal{M}_n) = \mathcal{K}_n^{\perp}.$$

Thus

$$\mathrm{Sol}_n \cap F_g = \mathcal{K}_n^{\perp} \cap F_g.$$

Restrict the dot product with kernel vectors to F_g:

$$\rho_g : \mathcal{K}_n \longrightarrow F_g^*, \qquad q \longmapsto (v \mapsto q \cdot v).$$

The kernel of ρ_g is exactly $\mathcal{K}_n \cap F_g^{\perp}$, so

$$\mathrm{rank}(\rho_g) = \delta_n - \eta_g(n).$$

Therefore the vectors in F_g orthogonal to all of $\mathcal{K}_n$ form a subspace of dimension

$$f_g(n) - \mathrm{rank}(\rho_g) = f_g(n) - \delta_n + \eta_g(n).$$

This proves the fixed-space formula, and Burnside's lemma gives the stated formula for $N_{\mathrm{Sol}}(n)$. □

Equivalently, one may sum over the five conjugacy classes of D_4. Let r be a quarter-turn rotation, u a half-turn rotation, a an axial reflection, and d a diagonal reflection. Then

$$N_{\mathrm{Sol}}(n) = \frac{1}{8}\left(2^{n^2-\delta_n} + 2 \cdot 2^{f_r-\delta_n+\eta_r} + 2^{f_u-\delta_n+\eta_u} + 2 \cdot 2^{f_a-\delta_n+\eta_a} + 2 \cdot 2^{f_d-\delta_n+\eta_d}\right),$$

where the f-values are

	e	r	u	a	d
n even	n^2	$n^2/4$	$n^2/2$	$n^2/2$	$n(n+1)/2$
n odd	n^2	$(n^2+3)/4$	$(n^2+1)/2$	$n(n+1)/2$	$n(n+1)/2.$

The corrections $\eta_e, \eta_r, \eta_u, \eta_a, \eta_d$ are evaluated at the same board size n, and $\eta_e = 0$. When $\delta_n = 0$, all correction terms vanish and this reduces to the formula for $N(n)$.

Proposition 26 (Asymptotics for solvable board orbits)**.** *As $n \to \infty$,*

$$N_{\mathrm{Sol}}(n) = 2^{n^2-\delta_n-3}\left(1 + O\left(2^{-(n^2-3n)/2}\right)\right).$$

In particular,

$$N_{\mathrm{Sol}}(n) \sim \frac{|\,\mathrm{Sol}_n\,|}{8}.$$

Proof. The identity contribution in the closed Burnside formula is

$$2^{n^2-\delta_n} = |\,\mathrm{Sol}_n\,|.$$

For every nonidentity symmetry g, we have

$$f_g(n) \le \frac{n(n+1)}{2} \qquad \text{and} \qquad \eta_g(n) \le \delta_n.$$

Also $\delta_n = \dim \Lambda_n \le n$, by the chasing-lights reduction to bottom-row representatives. Hence each nonidentity contribution is at most

$$2^{n(n+1)/2-\delta_n+\eta_g(n)} \le 2^{n(n+1)/2},$$

and its ratio to the identity contribution is at most

$$2^{n(n+1)/2-(n^2-\delta_n)} \le 2^{-(n^2-3n)/2}.$$

There are only seven nonidentity group elements, so the stated asymptotic follows. □

6 Randomness

6.1 Random clicking and hypercube mixing

There is a natural random process on solvable boards: begin with the dark board and repeatedly click a uniformly chosen button. For total-variation mixing it is convenient to use the standard aperiodic version of this walk. Let $N = n^2$, and from a board $b \in \mathrm{Sol}_n$ choose uniformly from the $N + 1$ moves

$$b, \qquad b + C_{i,j} \quad ((i, j) \in [n] \times [n]).$$

In other words, at each step we either do nothing or click one of the N buttons, all with probability $1/(N + 1)$. The uniform distribution on Sol_n is stationary, since this is a symmetric random walk on the abelian group Sol_n with generating set $\{C_{i,j} \mid (i, j) \in [n] \times [n]\}$. This stationary distribution is not the uniform distribution on the full matrix space $M_{n\times n}(\mathbb{Z}_2)$, unless every board is solvable. Viewed as a probability measure on $M_{n\times n}(\mathbb{Z}_2)$, it gives mass $1/|\,\mathrm{Sol}_n\,|$ to each solvable board and mass 0 to every unsolvable board. The random-click walk starts at the dark board and never leaves Sol_n.

Let

$$\phi : \mathbb{Z}_2^N \longrightarrow \mathrm{Sol}_n, \qquad \phi(v) = \mathcal{M}_n v.$$

The columns of $\mathcal{M}_n$ record the effect of the N coordinate clicks. Thus the random-click walk on Sol_n is the image under ϕ of the aperiodic coordinate-flip walk on the N-dimensional hypercube $\mathbb{Z}_2^N$. When $\mathcal{M}_n$ has full rank, this is literally the usual hypercube walk in the click basis. When $\mathcal{M}_n$ has a kernel, it is a quotient of that hypercube walk.

Proposition 27 (Random clicking mixes in cube time)**.** *Fix $0 < \varepsilon < 1$, and let $t^{(n)}_{\mathrm{mix}}(\varepsilon)$ denote the total-variation mixing time of the aperiodic random-click walk on Sol_n, with stationary distribution uniform on Sol_n. Then*

$$t^{(n)}_{\mathrm{mix}}(\varepsilon) = O(n^2 \log n).$$

More precisely, if $N = n^2$, then

$$t^{(n)}_{\mathrm{mix}}(\varepsilon) \le C_\varepsilon N \log N$$

for a constant C_ε independent of n. If $\mathcal{M}_n$ has full rank, then the walk has the same order of mixing as the N-cube:

$$t^{(n)}_{\mathrm{mix}}(\varepsilon) = \Theta(N \log N) = \Theta(n^2 \log n).$$

Proof. Let $\mathbf{X}_t$ be the aperiodic coordinate-flip walk on $\mathbb{Z}_2^N$, started at zero, and let $\mathbf{Y}_t = \phi(\mathbf{X}_t)$. By construction, $\mathbf{Y}_t$ has exactly the same law as the random-click walk on Sol_n. Its stationary measure is therefore supported only on Sol_n. Also, the image of the uniform distribution on $\mathbb{Z}_2^N$ under ϕ is the uniform distribution on Sol_n, because all fibers of the linear map ϕ have the same cardinality.

Total variation distance cannot increase under a deterministic map. Hence

$$\|\mathcal{L}(\mathbf{Y}_t) - U_{\mathrm{Sol}_n}\|_{\mathrm{TV}} \le \left\|\mathcal{L}(\mathbf{X}_t) - U_{\mathbb{Z}_2^N}\right\|_{\mathrm{TV}}.$$

The Diaconis–Shahshahani Fourier method for random walks on finite groups [DS81, DS87], applied in the hypercube setting by Diaconis, Graham, and Morrison [DGM90], gives mixing time $\Theta(N \log N)$ for this aperiodic random walk on the N-cube. The displayed inequality therefore gives

$$t^{(n)}_{\mathrm{mix}}(\varepsilon) \le C_\varepsilon N \log N = O(n^2 \log n).$$

If $\mathcal{M}_n$ has full rank, then ϕ is an isomorphism $\mathbb{Z}_2^N \cong \mathrm{Sol}_n$, so the random-click walk is exactly the hypercube walk after relabeling the coordinates. In that case the matching lower bound is also inherited from the hypercube. □

The laziness in this subsection is not merely technical. If one clicks exactly one button at every integer time, then the walk has a parity obstruction and does not converge to the uniform distribution in ordinary total variation on Sol_n. Indeed, the all-ones board lies in Sol_n, so there is a vector a with $\mathcal{M}_n a = \mathbf{1}$. The linear functional $b \mapsto a \cdot \widehat{b}$ takes the value 1 on every click matrix $C_{i,j}$, and therefore flips at every non-lazy step. The non-lazy walk alternates between two parity classes. The lazy walk above, or equivalently a continuous-time random-click walk, removes this periodicity while preserving the $n^2 \log n$ scale.

6.2 Random boards after chasing

Let $\mathbf{B}$ be chosen uniformly from the full board space $M_{n\times n}(\mathbb{Z}_2)$, meaning that the entries of $\mathbf{B}$ are independent fair bits. Since $\mathrm{CL} : M_{n\times n}(\mathbb{Z}_2) \to L_n$ is a surjective linear map, the induced distribution of $\mathrm{CL}(\mathbf{B})$ on L_n is uniform. Thus every bottom row occurs with probability 2^{-n}, and

$$\Pr(\mathrm{wt}(\mathrm{CL}(\mathbf{B})) = k) = 2^{-n}\binom{n}{k}.$$

Proposition 28 (Distribution after chasing)**.** *If* $\mathbf{S}$ *is uniform on* Sol_n*, then* $\mathrm{CL}(\mathbf{S})$ *is uniform on* $\mathcal{C}_n$*.*

Proof. Restrict CL to Sol_n. If $s \in \mathrm{Sol}_n$, then $\mathrm{CL}(s)$ differs from s by a sum of click matrices, so $\mathrm{CL}(s) \in \mathrm{Sol}_n$; also $\mathrm{CL}(s) \in L_n$. Hence $\mathrm{CL}(\mathrm{Sol}_n) \subseteq L_n \cap \mathrm{Sol}_n$. Conversely, every $\ell \in L_n \cap \mathrm{Sol}_n$ is fixed by the chasing algorithm, so $\ell = \mathrm{CL}(\ell)$. Thus $\mathrm{CL}(\mathrm{Sol}_n) = \mathcal{C}_n$. A linear map between finite vector spaces has fibers of equal size over its image, so the pushforward of the uniform distribution on Sol_n is uniform on $\mathcal{C}_n$. □

The event $\mathbf{B} \in \mathrm{Sol}_n$ is therefore equivalent to $\mathrm{CL}(\mathbf{B}) \in \mathcal{C}_n$. Since $\mathrm{CL}(\mathbf{B})$ is uniform on L_n,

$$\Pr(\mathbf{B} \in \mathrm{Sol}_n) = \frac{|\mathcal{C}_n|}{|L_n|} = \frac{1}{|\Lambda_n|} = 2^{-\delta_n},$$

where δ_n is given by the rank-deficiency criterion. In this form, solvability is read from the uniform bottom-row representative: the solvable boards are exactly those whose chased bottom row lands in the subspace $\mathcal{C}_n$.

Proposition 29 (Even-coordinate projection)**.** *Let*

$$\pi_e : \mathbb{Z}_2^{2n+1} \longrightarrow \mathbb{Z}_2^n, \qquad (x_1, \ldots, x_{2n+1}) \longmapsto (x_2, x_4, \ldots, x_{2n})$$

be projection to the even coordinates. Then

$$\pi_e(\mathcal{C}_{2n+1}) = \mathcal{C}_n.$$

Consequently, if $\mathbf{S}$ *is uniform on* Sol_{2n+1}*, then the even-coordinate part of* $\mathrm{CL}(\mathbf{S})$ *is uniform on* $\mathcal{C}_n$*.*

Proof. Order the coordinates of K_{2n+1} as

$$1, 3, \ldots, 2n+1, 2, 4, \ldots, 2n.$$

By the block decomposition in the chasing-lights section,

$$K_{2n+1} \sim \begin{pmatrix} H_n & J_n K_n \\ K_n J_n^T & K_n \end{pmatrix}.$$

Projecting the row space to the even coordinates gives the row space generated by the rows of $J_n K_n$ and K_n. Each row of $J_n K_n$ is a sum of adjacent rows of K_n, so it already lies in $\mathrm{rowspan}(K_n)$. The lower-right block contributes all rows of K_n. Hence the projected row space is exactly $\mathrm{rowspan}(K_n) = \mathcal{C}_n$. The final statement is again the equal-fiber property for a surjective linear map, applied to $\pi_e : \mathcal{C}_{2n+1} \to \mathcal{C}_n$. □

Thus the family tree for the effect matrices also has a probabilistic interpretation: along $K_n \mapsto K_{2n+1}$, the old bottom-row subspace is recovered as the even-coordinate marginal of the new one, while the off-diagonal block $J_n K_n$ controls how adjacent-row sums of the old subspace couple to the new odd coordinates.

6.3 Endpoint thresholds on the family $n = 5m - 1$

Let $\mathbf{B}_{n,p}$ be the random board whose entries are independent Bernoulli p bits. On the infinite deficient family $n = 5m-1$, there is no nonzero constant critical probability below which solvability holds with high probability. The obstruction already appears in the four-dimensional kernel coming from the 4×4 board.

Let

$$\theta_m : [5m - 1] \longrightarrow \{0, 1, 2, 3, 4\}$$

be the folded sequence

$$1, 2, 3, 4, 0, 4, 3, 2, 1, 0, 1, 2, 3, 4, 0, \ldots$$

truncated after $5m - 1$ terms. If $q \in \ker(\mathcal{M}_4)$, extend its indices by setting $q_{0,a} = q_{a,0} = 0$, and define $E_m(q) \in M_{(5m-1)\times(5m-1)}(\mathbb{Z}_2)$ by

$$E_m(q)_{i,j} = q_{\theta_m(i),\theta_m(j)}.$$

Lemma 30. *For every $q \in \ker(\mathcal{M}_4)$, one has*

$$E_m(q) \in \ker(\mathcal{M}_{5m-1}).$$

The map $E_m : \ker(\mathcal{M}_4) \to \ker(\mathcal{M}_{5m-1})$ is injective, and

$$\mathrm{wt}(E_m(q)) = m^2 \, \mathrm{wt}(q).$$

Proof. The sequence θ_m folds each copy of $[4]$ across a zero separator. If $\theta_m(i) = a \in [4]$, then the two neighboring θ_m-values are the two path-neighbors of a, with the missing boundary neighbor recorded as 0. If $\theta_m(i) = 0$, then the two neighboring values are equal, so their contributions cancel over $\mathbb{Z}_2$. The same statement holds in the column coordinate. Therefore the cross-neighborhood sum of $E_m(q)$ at (i, j) is exactly the corresponding cross-neighborhood sum of q on the 4×4 board, interpreted with zero boundary values. Since $q \in \ker(\mathcal{M}_4)$, this sum is zero.

Each value $1, 2, 3, 4$ occurs exactly m times in the sequence θ_m. Hence every nonzero entry of q is repeated on an $m \times m$ set of entries of $E_m(q)$, proving the weight formula. The same formula implies injectivity. □

Proposition 31 (No constant threshold on $n = 5m - 1$)**.** *Let*

$$P_m(p) = \Pr(\mathbf{B}_{5m-1,p} \in \mathrm{Sol}_{5m-1}).$$

Then

$$P_m(p) \le \frac{1}{16}\left(1 + 13(1-2p)^{8m^2} + 2(1-2p)^{12m^2}\right).$$

Consequently, for every fixed $0 < p < 1$,

$$\limsup_{m\to\infty} P_m(p) \le \frac{1}{16}.$$

In particular, there is no constant $p_0 > 0$ such that $p < p_0$, or symmetrically $p > 1 - p_0$, forces solvability with high probability along this family.

Proof. Choose a basis $q_1, \ldots, q_4$ for $\ker(\mathcal{M}_4)$, and let

$$\Psi_m(b) = \left(\langle \widehat{b}, \widehat{E_m(q_1)}\rangle, \ldots, \langle \widehat{b}, \widehat{E_m(q_4)}\rangle\right) \in \mathbb{Z}_2^4.$$

If $\mathbf{B}_{5m-1,p}$ is solvable, then it is orthogonal to every vector in $\ker(\mathcal{M}_{5m-1})$, so $\Psi_m(\mathbf{B}_{5m-1,p}) = 0$. Thus

$$P_m(p) \le \Pr(\Psi_m(\mathbf{B}_{5m-1,p}) = 0).$$

Fourier inversion on $\mathbb{Z}_2^4$ gives

$$\Pr(\Psi_m(\mathbf{B}_{5m-1,p}) = 0) = 2^{-4} \sum_{q\in\ker(\mathcal{M}_4)} (1-2p)^{\mathrm{wt}(E_m(q))}.$$

A direct computation in the four-dimensional space $\ker(\mathcal{M}_4)$ shows that it contains one vector of weight 0, thirteen vectors of weight 8, and two vectors of weight 12. Using the weight formula from the lemma gives the displayed bound. If $0 < p < 1$ is fixed, then $|1-2p| < 1$, so the two nonconstant terms tend to zero. □

Thus the only high-probability regime near the endpoints is the sparse one. Put

$$r_m = \min(p_m, 1-p_m), \qquad n_m = 5m - 1.$$

If

$$r_m n_m^2 \longrightarrow 0,$$

then $\mathbf{B}_{n_m,p_m}$ is, with high probability, either the zero board or the all-ones board; both are solvable. Conversely, if $r_m n_m^2$ stays bounded away from 0 along a subsequence, then the preceding four obstructions bound the solvability probability away from 1 along that subsequence. Therefore the threshold scale is $r_m \asymp n_m^{-2}$, not a positive constant probability.

6.4 Bottom-row linear obstructions

Let

$$\mathcal{C}_n^\perp = \{h \in L_n \mid h \cdot c = 0 \text{ for every } c \in \mathcal{C}_n\}$$

be the orthogonal complement of the solvable bottom-row subspace.

Proposition 32 (Bottom-row obstruction test)**.** *Let $h_1, \ldots, h_d$ be a basis for $\mathcal{C}_n^\perp$. A board $b \in M_{n\times n}(\mathbb{Z}_2)$ is solvable if and only if*

$$h_i \cdot \mathrm{CL}(b) = 0 \qquad (1 \le i \le d).$$

Proof. A board b is solvable if and only if its chased representative $\mathrm{CL}(b)$ lies in $L_n \cap \mathrm{Sol}_n = \mathcal{C}_n$. A vector of L_n lies in $\mathcal{C}_n$ if and only if it is orthogonal to every vector in a basis of $\mathcal{C}_n^\perp$. □

References

[AF98] Marlow Anderson and Todd Feil. Turning lights out with linear algebra. *Mathematics Magazine*, 71(4):300–303, 1998.

[BBH21] Abraham Berman, Franziska Borer, and Norbert Hungerbühler. Lights out on graphs. *Mathematische Semesterberichte*, 68:237–255, 2021.

[BR96] Rana Barua and S. Ramakrishnan. σ-game, σ^+-game and two-dimensional additive cellular automata. *Theoretical Computer Science*, 154(2):349–366, 1996.

[Bro] A. E. Brouwer. The all-ones problem. https://www.win.tue.nl/~aeb/ca/madness/madrect.html. Accessed May 19, 2026.

[DGM90] Persi Diaconis, Ronald L. Graham, and John A. Morrison. Asymptotic analysis of a random walk on a hypercube with many dimensions. *Random Structures & Algorithms*, 1(1):51–72, 1990.

[DS81] Persi Diaconis and Mehrdad Shahshahani. Generating a random permutation with random transpositions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 57(2):159–179, 1981.

[DS87] Persi Diaconis and Mehrdad Shahshahani. Time to reach stationarity in the bernoulli–laplace diffusion model. *SIAM Journal on Mathematical Analysis*, 18(1):208–218, 1987.

[FY13] Rudolf Fleischer and Jiajin Yu. A survey of the game "lights out!". In *Space-Efficient Data Structures, Streams, and Algorithms*, volume 8066 of *Lecture Notes in Computer Science*, pages 176–198. Springer, Berlin, Heidelberg, 2013.

[GKW02] John L. Goldwasser, William F. Klostermeyer, and Henry Ware. Fibonacci polynomials and parity domination in grid graphs. *Graphs and Combinatorics*, 18(2):271–283, 2002.

[GWW11] John Goldwasser, Xinmao Wang, and Yaokun Wu. Minimum light numbers in the σ-game and lit-only σ-game on unicyclic and grid graphs. *The Electronic Journal of Combinatorics*, 18(1):P214, 2011.

[HMP04] Markus Hunziker, Antonio Machiavelo, and Jihun Park. Chebyshev polynomials over finite fields and reversibility of σ-automata on square grids. *Theoretical Computer Science*, 320(2–3):465–483, 2004. https://reader.elsevier.com/reader/sd/pii/S0304397504001823.

[Hua15] Hau-Wen Huang. Lit-only σ-game on nondegenerate graphs. *Journal of Algebraic Combinatorics*, 41:385–395, 2015.

[KG02] William F. Klostermeyer and John L. Goldwasser. Nullspace-primes and fibonacci polynomials. *The Fibonacci Quarterly*, 40(4):323–327, 2002. https://www.fq.math.ca/Scanned/40-4/klostermeyer.pdf.

[Kre17] Martin Kreh. "lights out" and variants. *The American Mathematical Monthly*, 124(10):937–950, 2017.

[Mat] Proving properties of matrices over $\mathbb{Z}_2$. https://www.researchgate.net/publication/257334572_Proving_properties_of_matrices_over_Z2. Accessed May 19, 2026.

[Nel26] Todd W. Neller. Top-row effect pattern images for *Lights Out*. Personal communication, Gettysburg College, 2026.

[Sch] Jaap Scherphuis. Lights out. https://www.jaapsch.net/puzzles/lights.htm. Accessed May 19, 2026.

[Sut90] Klaus Sutner. The σ-game and cellular automata. *The American Mathematical Monthly*, 97(1):24–34, 1990.

[Tor11] Bruce Torrence. The easiest lights out games. *The College Mathematics Journal*, 42(5):361–372, 2011.

[Wola] Wolfram MathWorld. Chebyshev polynomial of the second kind. https://mathworld.wolfram.com/ChebyshevPolynomialoftheSecondKind.html. Accessed May 19, 2026.

[Wolb] Wolfram MathWorld. Lights out puzzle. https://mathworld.wolfram.com/LightsOutPuzzle.html. Accessed May 19, 2026.